

# Ranking via Arrow-Debreu Equilibrium

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## Abstract

In this paper, we establish a connection between ranking theory and general equilibrium theory. First of all, we show that the ranking vector of PageRank or Invariant method is precisely the equilibrium of a special Cobb-Douglas market. This gives a natural economic interpretation for the PageRank or Invariant method. Furthermore, we propose a new ranking method, the *CES ranking*, which is *minimally fair*, *strictly monotone* and *invariant to reference intensity*, but not *uniform* or *weakly additive*.

## 1 Introduction

Ranking, which aggregates the preferences of individual agents over a set of alternatives, is not only a fundamental problem in social choice theory but also has many applications in real life. For instance, the well-known PageRank algorithm [7] is designed to rank Web pages while the Invariant Method [3, 5] is proposed to evaluate the intellectual influence of academic journals and papers.

Intuitively, the PageRank and the Invariant method share a common property in that the more “vote” an agent gets, the higher ranking he has. Although they work very well in practice, the economic interpretations of their effectiveness are not obvious. Slutski and Volji [3], as well as Palacios-Huerta and Volji [5], gave the first set of axioms that characterize the Invariant method. Later, Altman and Tennenholtz [6] gave a set of combinatorial axioms to characterize the PageRank algorithm, while Brandt and Fischer [4] interpreted PageRank as a solution of a weak tournament.

General equilibrium theory [9] is one of the most prominent theories in mathematical economics. It studies how a market system, known as the “invisible hand”, makes the demands of a market’s participants equal to its supplies. Arrow and Debreu [10] showed that under mild conditions, a market always has an equilibrium. The result of this research became known as the *Arrow-Debreu equilibrium*.

In this paper, we will establish a connection between ranking methods and the Arrow-Debreu equilibrium. Naturally, the preference of one agent for another, which is usually represented as a directed edge in a graph, can be viewed as the demand between agents. Intuitively, the more demands a good gets, the higher price it should have. Therefore, an equilibrium price could be a good candidate for a ranking vector. On the other hand, the PageRank and the Invariant method are the stationary distributions of ergodic Markov chains. Both the existence of a PageRank or an Invariant ranking vector and the existence of Arrow-Debreu equilibrium can be shown via the Brouwer's fixed point theories [10, 9]. We will interpret one form of a fixed point as the other. More specifically, we will show that the ranking vector of the PageRank or the Invariant method is indeed the equilibrium of a special Cobb-Douglas market. To the best of our knowledge, this is the first connection between ranking methods and the general equilibrium theory. Based on our observations, we propose a new ranking method, the CES ranking, which is minimally fair, strictly monotone, and invariant to reference intensity, but not uniform or weakly additive.

## 2 Preliminaries

### 2.1 The Ranking Problem

In this subsection, we will follow [6] to define *ranking problems*. Let  $A$  be a finite set, representing the set of agents, and  $M$  be a  $|A| \times |A|$  matrix, representing the preference relationships among the agents. A ranking problem is represented as  $\langle A, M \rangle$ . For any  $n \in \mathbb{N}$ , let  $\Delta_n = \{(x_1, \dots, x_n) | \forall i, x_i \geq 0 \text{ and } \sum_i x_i = 1\}$ .

**Definition 1** A ranking function maps a ranking problem  $\langle A, M \rangle$  to a vector  $\pi \in \Delta_{|A|}$ .

### 2.2 Markov chain and PageRank

**Markov chain** [8] A discrete *Markov chain* is a random process  $\{X_i\}$  on a state space  $S = \{s_1, \dots, s_n\}$  that satisfies the *Markov property*:

$$P(X_j | X_i, \dots, X_0) = P(X_j | X_i) = p_{ij},$$

where  $p_{ij}$  is the transition property from state  $s_i$  to  $s_j$  and  $\forall i, \sum_j p_{ij} = 1$ . Let  $P$  be the transition matrix. Correspondingly, we can define

**Definition 2 (State Transition Graph:)** Given a discrete Markov chain  $\{X_i\}$ , the corresponding state transition graph is  $G = (V, E, W)$  where  $V = S$  and  $(s_i, s_j) \in E$  iff  $p_{ij} > 0$ , and  $w_{s_i, s_j} = p_{ij}$ .

It is well known that if the state transition graph is strongly connected and aperiodic, the corresponding Markov chain is called *ergodic* and has a unique *stationary distribution*  $\pi$  such that

$$\pi = P^T \pi$$

**PageRank** [7] Let  $G = (V, E)$  be a directed graph with vertex set  $V$  and edge set  $E$ . We assume that there is no self-loop in  $G$ . Let  $N = |V|$ , and for a vertex  $i \in V$ , denote by  $\text{out}(i)$  the out-degree of  $i$ . The *transition matrix* of  $G$  is  $T = [T_{ij}]_{1 \leq i, j \leq N}$ :

$$T_{ij} = \begin{cases} \frac{1}{\text{out}(i)} & \text{if } (i, j) \in E \\ 0, & \text{otherwise} \end{cases}$$

Denote by  $e \in \mathbb{R}^N$  the all 1 row vector  $(1, 1, \dots, 1)$ , and by  $E \in \mathbb{R}^{N \times N}$  the all 1 matrix. Let  $\bar{T}$  be identical to  $T$  except that if a row in  $P$  is all 0, it should be replaced by  $e/N$ . A page without outgoing links is called a *dangling* page. For some constant  $c$ ,  $0 < c < 1$ , the transition matrix for the PageRank Markov chain is

$$P = c\bar{T} + (1 - c)E/N.$$

The PageRank  $\pi$  is the stationary distribution, i.e.,  $\pi P = \pi$ , of the above Markov chain  $P$ .

**Invariant Method** [3] In the definition of PageRank [7], if the transition matrix  $T$  is irreducible (the corresponding graph is strongly connected), its unique stationary distribution is the ranking vector. Thus, the PageRank and the Invariant method are essentially equivalent in mathematics.

### 2.3 Arrow-Debreu equilibrium of exchange markets

In an exchange market, there are  $m$  traders and  $n$  divisible goods. Let  $\mathcal{T} = \{T_1, \dots, T_m\}$  be the set of traders and  $\mathcal{G} = \{G_1, \dots, G_n\}$  be the set of goods. Each trader  $i$  has an initial endowment of  $w_{i,j} \geq 0$  of good  $j$  and a utility function  $u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}$ . The individual goal of trader  $T_i$  is to obtain a new bundle of goods that maximizes his utility. Let  $\mathbf{x}_i \in \mathbb{R}_+^n$  be the bundle of goods of  $T_i$  after the exchange, where  $x_{i,j}$  is the amount of good  $j$ . Naturally, the demand cannot exceed the supply:  $\sum_i x_{i,j} \leq \sum_i w_{i,j}$ , for every good  $j$ .

We use  $\mathbf{p} \in \Delta_n$  to denote a price vector, where  $p_j$  is the price of  $G_j$ . For any trader  $T_i$ , given  $\mathbf{p}$ , we let  $\mathbf{x}_i^*(\mathbf{p})$  denote the bundle of goods that maximize his utility under the budget constraint:

$$\mathbf{x}_i^*(\mathbf{p}) = \operatorname{argmax}_{\mathbf{x} \in \mathbb{R}_+^n, \mathbf{x} \cdot \mathbf{p} \leq \mathbf{w}_i \cdot \mathbf{p}} u_i(\mathbf{x}).$$

**Definition 3 (Arrow-Debreu equilibrium)** A market equilibrium is a price vector  $\mathbf{p} \in \Delta_n$  such that the market clears:

$$\begin{aligned} & \text{For every good } G_j \in \mathcal{G}, \sum_{i \in [m]} x_{i,j}^*(\mathbf{p}) \leq \sum_{i \in [m]} w_{i,j}; \text{ If } p_j > 0, \text{ then} \\ & \sum_{i \in [m]} x_{i,j}^*(\mathbf{p}) = \sum_{i \in [m]} w_{i,j}. \end{aligned}$$

### 2.4 CES Utility Functions

**CES utility functions:** [2, 9] The CES (Constant Elasticity of Substitution) function over a bundle of goods  $(x_{i1}, \dots, x_{in})$  is the family of utility functions  $u_i(x_{i1}, \dots, x_{in}) = (\sum_{j=1}^n \alpha_{ij} x_{ij}^{\rho_i})^{1/\rho_i}$ , where  $-\infty < \rho_i < 1$ ,  $\rho_i \neq 0$  and  $\alpha_{ij} \geq 0$ .

The parameter  $1/(1-\rho_i)$  is called the *elasticity of substitution*. The CES utility function has a very nice property: its demand functions have explicit forms. That is, given a strictly positive price vector  $\pi \in \mathbb{R}_{++}^n$ , the demand  $x_{ij}$  is

$$x_{ij} = \frac{\alpha_{ij}^{1/(1-\rho_i)}}{\pi_j^{1/(1-\rho_i)}} \times \frac{\sum_k \pi_k w_{ik}}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} \pi_k^{-\rho_i/(1-\rho_i)}}$$

There are three important utility functions within the CES category.

1.  $\rho_i \rightarrow 1$  corresponds to linear utility functions, where  $u_i(x_{i1}, \dots, x_{in}) = \sum_j \alpha_{ij} x_{ij}$ . In this case, the set of goods that the agent wants are perfect substitutes for each other.
2.  $\rho_i \rightarrow -\infty$  corresponds to Leontief utility functions. The Leontief utility function, in general, has the form of  $u_i(x_{i1}, \dots, x_{in}) = \min_{j: \beta_{ij} > 0} \frac{x_{ij}}{\beta_{ij}}$ , where  $\beta_{ij} \geq 0$ . In this case, the set of goods that the agent wants are perfect complement of each other.
3.  $\rho_i \rightarrow 0$  corresponds to Cobb-Douglas utility functions. The Cobb-Douglas utility function, in general, has the form of  $u_i(x_{i1}, \dots, x_{in}) = \prod_j x_{ij}^{\beta_{ij}}$ , where  $\beta_{ij} \geq 0$ . This demand function is a perfect balance of substitution and complementarity effects [2].

### 3 PageRank/Invariant Method V.S. a Cobb-Douglas Market

In this section, we will establish a connection between PageRank/Invariant Method and Arrow-Debreu equilibrium. We do this by showing a more general theorem about Markov chains.

**Theorem 1** *Given an ergodic Markov chain, there is a mapping from the Markov chain to a Cobb-Douglas market, such that the stationary distribution of the Markov chain is precisely the Arrow-Debreu equilibrium of the Cobb-Douglas market.*

**Proof.** The general idea of the proof is to reduce the state transition graph of an ergodic Markov chain to the economy graph of a Cobb-Douglas market. Given the state transition graph  $G = (V, E, W)$  and the transition matrix  $P$ , we can construct a Cobb-Douglas economy graph as follows: for  $i \in [1..n]$ , there is a trader  $T_i$  corresponding to each state  $s_i$ , and there is a directed link from  $T_i$  to  $T_j$  iff  $(s_i, s_j) \in E$ ; for trader  $T_i$ , let  $N(T_i)$  be the set of outgoing neighbors of  $T_i$ . The utility function of  $T_i$  is  $u_i(x_i) = \prod_{j \in N(T_i)} x_{ij}^{p_{ij}}$ , where  $p_{ij}$  is the transition probability. Initially  $T_i$  has one unit of the good  $G_i$  but no other goods. We call such a Cobb-Douglas economy  $M$ . We claim that the market equilibrium of  $M$  is also the stationary distribution of  $P$ .

First of all, since  $G$  is strongly connected and the Cobb-Douglas utility function belongs to the CES utility function class, according to Theorem 1 of Codenotti *et al.* [2]  $M$  has a strictly positive equilibrium. By the demand function of CES utility function, when  $\rho_i \rightarrow 0$ ,

$$x_{ij} = \frac{p_{ij}\pi_i}{\pi_j}.$$

By the definition of Arrow-Debreu equilibrium, for every good with strictly positive price, its demand must be equal to its supply. Thus,

$$\sum_i x_{ij} = \sum_i \frac{p_{ij}\pi_i}{\pi_j} = 1.$$

Equivalently,

$$\sum_i p_{ij}\pi_i = \pi_j.$$

Thus,  $\pi$  is the stationary distribution of the Markov chain  $P$ . ■

Actually, by the above reduction, we also implicitly show that  $M$  has a unique equilibrium. Most importantly, since the PageRank or Invariant method is a special Markov chain, the ranking vector of the PageRank or the Invariant method can also be interpreted as the equilibrium of a Cobb-Douglas market.

**Remarks:** Eaves [15] showed that the computation of an equilibrium for a Cobb-Douglas market can be reduced to solving a linear equation system. Although it was not explicitly claimed in [15], Eaves's result implies that an equilibrium of a Cobb-Douglas market actually corresponds to a principle eigenvector of a stochastic matrix. In Theorem 1, we show the reverse direction of Eaves' reduction. That is a principle eigenvector of a stochastic matrix corresponds to an equilibrium of a special Cobb-Douglas market. Thus, our result, which is essentially different from Eaves' in terms of motivations, is a complement of the result in [15].

**Economic Interpretations:** It is believed that the validity of PageRank comes from the fact that the Markov chain is a good model for the Web surfing behavior of Web users. In web graph, a link from page  $p$  to page  $q$  means that a Web user at page  $p$  may find the content of page  $q$  is useful. Thus, a link in web graph means a vote or reference. Intuitively, the more votes a page gets, the more important it is. Indeed, for a Web user, his goal is to maximize his information needs by following outgoing links of a page to visit other pages. Thus, in our Cobb-Douglas economy graph, each Web page is corresponding to an agent, the content of the page is corresponding to the good the agent initially owns, and a link from  $p$  to  $q$  means that the agent on page  $p$  has a demand for the content of page  $q$ . Intuitively, the more "demand" a page gets, the more important it is.

Theorem 1 provides a new perspective to view PageRank. That is the substitution and complementarity effects of outgoing links. For instance, suppose

we have a directory page of a university, which has outgoing links pointing to the home pages of each of the university's departments. If a Web user clicks one of the outgoing links, it is unlikely that he will click any other. Thus, for this page, its outgoing links are more likely to be substitutes for each other than complements. On the other hand, suppose we have a news page, which has outgoing links pointing to related news pages. A Web user who clicks one of the outgoing links is likely to click another one. Thus, for this page, its outgoing links are more likely to be complements for each other than substitutes. By Theorem 1 and the properties of the Cobb-Douglas utility function, the set of pages that a Web page points to is a mix of the substitution and complementarity effects with elasticity of substitution 1 in PageRank.

## 4 Ranking via Arrow-Debreu Equilibrium

The Cobb-Douglas utility function corresponds to the CES utility function with  $\rho \rightarrow 0$ . Thus, by choosing CES utility functions with different elasticities, we can naturally extend the idea of PageRank to a new spectrum of ranking algorithms. We propose the ranking method, which is called *CES ranking*, below.

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### Algorithm 1 CES ranking

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1. Given the agents set  $A$ , choose a CES utility function  $u_i$  for each agent  $i \in A$  and set the initial endowment  $w_i$  of it as  $\forall j \neq i, w_{ij} = 0$  but  $w_{ii} = 1$ . The new ranking problem is  $\langle A, \{\alpha_{ij}\}_{i \in A, j \in A}, \{\rho_i\}_{i \in A}, \{w_i | i \in A\} \rangle$ . W.L.O.G., for any  $i$ , let  $\sum_j \alpha_{ij} = 1$ .
  2. If for agent  $i$ ,  $\alpha_{ij} = 0$  for all  $j$ , set  $\alpha_{ij} = 1/|A|$  for each  $j$ . Hence,  $u_i = (\sum_{j=1}^n (1/|A|) x_{ij}^{\rho_i})^{1/\rho_i}$ .
  3. For each agent  $i$ , update  $\alpha_{ij}$  to be  $\alpha_{ij} * \beta + (1/|A|) * (1 - \beta)$  for every  $j$ , where  $\beta = 0.85$ . Correspondingly, the updated utility function is  $u_i = (\sum_{j=1}^n (\alpha_{ij} * \beta + (1/|A|) * (1 - \beta)) x_{ij}^{\rho_i})^{1/\rho_i}$ .
  4. Construct the economy graph  $G$  with respect to the CES economy defined above. It is easy to see that  $G$  is strongly connected.
  5. Compute an equilibrium of  $G$  and use it as the ranking vector.
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Note that the new CES ranking problem  $\langle A, \{\alpha_{ij}\}_{i \in A, j \in A}, \{\rho_i\}_{i \in A}, \{w_i | i \in A\} \rangle$  is a generalization of the ranking problem  $\langle A, M \rangle$  defined in the Preliminaries. Now we discuss the existence, uniqueness, and efficiency of the CES ranking, as well as some other properties related to ranking.

**Existence of a ranking vector:** By Theorem 1 in [2], as long as the economy graph  $G$  is strongly connected, there is always a strictly positive equilibrium. By the definition of the CES ranking, it is obvious that the economy graph of it is strongly connected. Thus, a ranking vector always exists.

**Uniqueness:** According to [2], for CES utility functions with  $-1 \leq \rho < 1$ , the set of equilibria is convex. We further show that:

**Claim 2** *The CES ranking has a unique ranking vector if  $\forall i, \rho_i \geq 0$ .*

In order to prove this claim, we first introduce the definition of gross substitute (GS).

**Definition 4** [9] For any  $j$ , let  $z_j = \sum_i x_{ij} - \sum_i w_{ij}$  be the excess demand function for  $G_j$ . The function  $z(\cdot)$  has the gross substitute property if whenever  $\pi'$  and  $\pi$  are such that, for some  $l$ ,  $\pi'_l > \pi_l$  and  $\pi'_j = \pi_j$  for  $j \neq l$ , we have  $z_j(\pi') > z_j(\pi)$  for  $j \neq l$ .

If in the above definition the inequalities are weak, the property is referred to as weak gross substitute (WGS). It is well known that:

**Theorem 3** [9] If the aggregate excess demand function of an exchange economy has the GS property, the economy has at most one equilibrium.

Now we can prove the above claim.

**Proof.**<sup>1</sup> In the CES ranking, by the Theorem 1 and Lemma 1 in [2], as long as the economy graph  $G$  is strongly connected, an equilibrium exists and every equilibrium is strictly positive. Suppose we have two equilibria  $\pi'$  and  $\pi$  such that for  $l$ ,  $\pi'_l > \pi_l$  and  $\pi'_h = \pi_h$  for  $h \neq l$ . Note that  $\forall i, j, \alpha_{ij} > 0$ . Thus, for any  $j \neq l$ ,

$$\begin{aligned} \sum_i x_{ij} &= \sum_i \frac{\alpha_{ij}^{1/(1-\rho_i)}}{\pi_j^{1/(1-\rho_i)}} \times \frac{\pi_i}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} \pi_k^{-\rho_i/(1-\rho_i)}} \\ &= \sum_i \alpha_{ij}^{1/(1-\rho_i)} \times \frac{\pi_i/\pi_j}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} (\pi_k/\pi_j)^{-\rho_i/(1-\rho_i)}} \\ &< \frac{\alpha_{lj}^{1/(1-\rho_l)} \times \pi'_l/\pi'_j}{\sum_k \alpha_{lk}^{1/(1-\rho_l)} (\pi'_k/\pi'_j)^{-\rho_l/(1-\rho_l)}} + \sum_{i \neq l} \frac{\alpha_{ij}^{1/(1-\rho_i)} \times \pi'_i/\pi'_j}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} (\pi'_k/\pi'_j)^{-\rho_i/(1-\rho_i)}} \\ &= \sum_i x'_{ij} \end{aligned}$$

Thus, in the CES ranking, the excess demand function has the GS property. Therefore, its equilibrium and the ranking vector are unique. ■

However, when  $\rho < -1$ , there may be multiple equilibria sets. Actually, for ranking problems, we do not have to insist on the uniqueness of ranking vectors. That is because in most cases, there are different ranking criteria for one ranking problem. It is not surprising that different criteria induce different rankings.

**Efficiency of Computation:** When  $-1 \leq \rho < 1$ , an equilibrium of a CES market can be computed via convex programming [2]. Thus, it is in polynomial time. However, for some special utility functions (such as the Leontief utility function), it is PPAD-hard [12] to compute an equilibrium of it. However, for

<sup>1</sup>It is well known [2] that the CES utility functions satisfy WGS when  $\rho \geq 0$ . However, WGS does not imply the uniqueness of equilibrium.

ranking problems that have relatively small sizes or do not require the real time computation of ranking vectors, the efficiency may not be a serious concern.

In the next section, we study five natural properties, which are satisfied by the PageRank and the Invariant method, with respect to the CES ranking. First, we extend the concept of minimally fair [1] to the CES ranking.

**Definition 5** *A ranking system is minimally fair if when for any  $i, j$ ,  $\alpha_{ij} = 0$ , the ranking score of agent  $i$  equals that of  $j$  for agent  $i, j \in A$ .*

**Claim 4** *The CES ranking is minimally fair.*

**Proof.** Since initially  $\alpha_{ij} = 0$  for any  $i, j$ , in order to make the economy graph strongly connected, we set the utility function for each agent as  $u_i = ((\sum_j \frac{1}{n} x_{ij}^\rho)^{1/\rho})$ . With this setup, by the market clearing condition, we get

$$\sum_i \frac{(1/n)^{1/(1-\rho)}}{\pi_j^{1/(1-\rho)}} \times \frac{\pi_i}{\sum_k (1/n)^{1/(1-\rho)} \pi_k^{-\rho/(1-\rho)}} = 1$$

Thus,  $\forall j$ ,

$$\pi_j = (1/(\sum_k \pi_k^{-\rho/(1-\rho)}))^{1-\rho}.$$

Note that the right-hand side of the above equation is independent of  $j$ . Thus,  $\pi = (1/n, \dots, 1/n)$  is the only equilibrium for the market. Therefore, the CES ranking is minimally fair. ■

Next, we extend the strictly monotone definition in [1] to the CES ranking.

**Definition 6** *A ranking system is strictly monotone iff for any two agents  $i$  and  $j$ , if for any other agent  $p$ ,  $\alpha_{pi} \leq \alpha_{pj}$  and there exists  $h$  such that  $\alpha_{hi} < \alpha_{hj}$ , the ranking score of agent  $i$  is strictly less than that of  $j$ .*

**Claim 5** *The CES ranking is strictly monotone when the utility functions of all the agents have the same elasticity of substitution.*

**Proof.** By the market clearing condition, for any two agents  $i$  and  $j$ ,

$$\begin{aligned} \pi_i^{1/(1-\rho)} &= \sum_p \frac{\alpha_{pi}^{1/(1-\rho)} \times \pi_p}{\sum_k \alpha_{pk}^{1/(1-\rho)} \pi_k^{-\rho/(1-\rho)}} \\ &< \sum_p \frac{\alpha_{pj}^{1/(1-\rho)} \times \pi_p}{\sum_k \alpha_{pk}^{1/(1-\rho)} \pi_k^{-\rho/(1-\rho)}} \\ &= \pi_j^{1/(1-\rho)} \end{aligned}$$

Thus,  $\pi_i < \pi_j$ . ■

Actually, the property of strictly monotone corresponds to the intuition that the more demands a good gets, the higher price it is.

Slutzki and Volij showed [3] that,



**Theorem 6** [3] *If a ranking system satisfies uniform, weakly additive and invariant to reference intensity, the ranking system must be the Invariant method.*

In the next section, we will study the relationship between the CES ranking and the three properties above.

**Definition 7** [3] *A ranking problem is regular if  $\forall i, j, \sum_k \alpha_{ik} = \sum_k \alpha_{jk}$  while  $\forall i, j, \sum_k \alpha_{ki} = \sum_k \alpha_{kj}$ . A ranking function is uniform if for every regular ranking problem, the ranking score of each agent is  $\frac{1}{N}$  where  $N$  is the number of agents.*

**Claim 7** *The CES ranking is not uniform.*

**Proof.** Suppose a system has three agents while the parameters of the agents are  $\rho_1 = \rho_2 = \rho_3 = \frac{1}{2}$  and  $\alpha_{11} = \frac{1}{3}, \alpha_{12} = \frac{1}{3}, \alpha_{13} = \frac{1}{3}, \alpha_{21} = \frac{5}{12}, \alpha_{22} = \frac{1}{6}, \alpha_{23} = \frac{5}{12}, \alpha_{31} = \frac{1}{4}, \alpha_{32} = \frac{1}{2}, \alpha_{33} = \frac{1}{4}$ . By the market clearing condition,

$$\pi_1^2 = \frac{\frac{1}{3}^2 \times \pi_1}{\frac{1}{3}^2 \pi_1^{-1} + \frac{1}{3}^2 \pi_2^{-1} + \frac{1}{3}^2 \pi_3^{-1}} + \frac{\frac{5}{12}^2 \times \pi_2}{\frac{5}{12}^2 \pi_1^{-1} + \frac{1}{6}^2 \pi_2^{-1} + \frac{5}{12}^2 \pi_3^{-1}} + \frac{\frac{1}{4}^2 \times \pi_3}{\frac{1}{4}^2 \pi_1^{-1} + \frac{1}{2}^2 \pi_2^{-1} + \frac{1}{4}^2 \pi_3^{-1}}$$

It is easy to check that  $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  cannot satisfy the above equation. Thus, the CES ranking is not uniform. ■

Actually, the uniform property requires that the ranking score of an agent is linearly proportional to the number of “votes” it gets. This assumption may not be reasonable universally. For instance, in the citation analysis, suppose both paper A and B have  $m$  citations. In an extreme case, the citations of paper A may only come from one research group while the citations of B come from different research groups. Intuitively, paper B should be more important than paper A. However, any ranking algorithm with the uniform property cannot distinguish those two cases. The CES ranking, as a nonlinear ranking method, may have the potential to find out new signals that were missed by the uniform ranking methods.

The *weakly additive* [3] property says that for a regular ranking problem, the ranking score is still linearly proportional to the “votes” after a symmetric perturbation. Since the CES ranking is not uniform, it cannot satisfy the weakly additive property, either.

**Definition 8** [3] *A ranking system is invariant to reference intensity if for any agent  $i$ , when we multiply  $\alpha_{ij}$  by a positive constant  $\lambda$  for every  $j$ , it cannot change the ranking score of any agent.*

**Claim 8** *The CES ranking is invariant to reference intensity.*

**Proof.** Note that

$$\begin{aligned}
x_{ij} &= \frac{\alpha_{ij}^{1/(1-\rho_i)}}{\pi_j^{1/(1-\rho_i)}} \times \frac{\sum_k \pi_k w_{ik}}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} \pi_k^{-\rho_i/(1-\rho_i)}} \\
&= \frac{(\lambda \alpha_{ij})^{1/(1-\rho_i)}}{\pi_j^{1/(1-\rho_i)}} \times \frac{\sum_k \pi_k w_{ik}}{\sum_k (\lambda \alpha_{ik})^{1/(1-\rho_i)} \pi_k^{-\rho_i/(1-\rho_i)}} \\
&= \frac{\alpha_{ij}^{1/(1-\rho_i)}}{\pi_j^{1/(1-\rho_i)}} \times \frac{\sum_k \pi_k w_{ik}}{\sum_k \alpha_{ik}^{1/(1-\rho_i)} \pi_k^{-\rho_i/(1-\rho_i)}}
\end{aligned}$$

Thus, multiplying  $\alpha_{ij}$  by a positive constant  $\lambda$  for every  $j$  cannot change the demand function. Therefore, the set of equilibria remains the same. ■

When we summarize the above claims together, we get

**Theorem 9** *The CES ranking is minimally fair, strictly monotone and invariant to reference intensity, but not uniform or weakly additive.*

## 5 Conclusion and Future Works

In this paper, we have established a connection between the ranking theory and the general equilibrium theory. First, we showed that the ranking vector of PageRank or Invariant method is actually the equilibrium of a special Cobb-Douglas market. This gives a natural economic interpretation for the PageRank and Invariant method. Furthermore, we propose a new ranking method, the CES ranking, which is minimally fair, strictly monotone, and invariant to reference intensity, but not uniform or weakly additive. The new CES ranking, compared to PageRank and the Invariant method, is nonlinear, and could be potentially used to find signals in a system missed by those existing ranking methods.

With the observations in this paper, we have a complete picture of the encoding power of the three limiting cases of CES utility functions. Pennock and Wellman [13] showed that economies with almost the linear utility functions can encode Bayesian networks. Codenotti *et al.* [12] proved that economies with the Leontief utility functions can encode bimatrix games. Now we demonstrate that economies with the Cobb-Douglas utility functions can encode Markov chains.

We believe that this paper points to a few promising directions that are worth further exploration.

- Explore more properties that the CES ranking satisfies and make justifications for the properties it does not satisfy.
- For various applications, what is the “right” utility function for each agent? We may go beyond the CES utility functions and explore other ones, such as WGS utility functions [2].

- Design efficient algorithms to compute ranking vectors.
- Further investigate the uniqueness of ranking vectors. If there are multiple equilibria points, do they induce the same ranking? If not, interpret their different economic meanings in the context of ranking.
- Last but not least, design an effective evaluation system for ranking methods and find an application where the CES ranking can outperform existing ranking methods.

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